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1977 J. Phys. A: Math. Gen. 10 325

(http://iopscience.iop.org/0305-4470/10/2/021)

J. Phys. A: Math. Gen., Vol. 10, No. 2, 1977. Printed in Great Britain.

COMMENT

Corrections to square lattice Potts model series

I G Enting

Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, Canberra, ACT, 2600, Australia

Received 31 August 1976

Abstract. The random-cluster formulation of the Potts model connects a number of lattice models and reveals a disagreement between series expansions published for special cases. The corrected series are given.

Fortuin and Kasteleyn (1972) first described the random-cluster model which they showed to be a generalization of the q-state Potts model (Potts 1952) with q no longer restricted to integers greater than one. Other special cases included the bond percolation problem (Essam 1972) as a $q \to 1$ limit and the q-colouring of a lattice (Nagle 1971). On the square lattice exact solutions are known for the Ising case $q = 2$ (Onsager 1944, Yang 1952) and for Potts models at the critical point (Potts 1952, Baxter 1973). The number of 3-colourings of the square lattice was obtained by Lieb (1967). Further investigation of these problems has depended largely on series expansions. These expansions have all been for special cases (with the exception of the work of Straley and Fisher 1973), partly because of the sheer complexity of working with three variable series and partly because important simplifications occur for $q = 1$ (Sykes and Glen 1976), $q = 2$ (Domb 1974) and for colouring problems (Baker 1971). There is, however, sufficient overlap to reveal discrepancies between some of the published series.

The Potts model can be defined in terms of variables t_i at each site of a square lattice. These variables can each be in one of q states. The energy E is made up of a contribution J for each pair of neighbours in different states and a contribution h for each site not in state '1'.

Using the variables

$$u = \exp(-J/kT) \tag{1}$$

$$\mu = \exp(-h/kT) \tag{2}$$

we can obtain low temperature (high field) expansions for the reduced partition function Λ:

$$\Lambda = \sum_{\text{states}} \exp(-E/kT) \tag{3}$$

or

$$\ln \Lambda = \sum_{n \geqslant 1} \mu^n L_n(u, q) = \sum_{m \geqslant 1} u^m \psi_m(\mu, q). \tag{4}$$

For zero field, $\mu = 1$, we have the expansion

$$\begin{aligned}\Lambda = 1 &+ (q-1)u^4 + 2(q-1)u^6 + (q-1)(2q-4)u^7 + (q-1)(-2q+9)u^8 \\ &+ (q-1)(12q-24)u^9 + (q-1)(6q^2-32q+54)u^{10} \\ &+ (q-1)(-10q^2+84q-128)u^{11} + (q-1)(q^3+55q^2-243q+302)u^{12} \\ &+ (q-1)(18q^3-178q^2+644q-720)u^{13} \\ &+ (q-1)(-37q^3+561q^2-1802q+1808)u^{14} \\ &+ (q-1)(8q^4+244q^3-1928q^2+5194q-4756)u^{15} \\ &+ (q-1)(58q^4-998q^3+6265q^2-15146q+12854)u^{16} \\ &+ (q-1)(2q^5-154q^4+3818q^3+\ldots)u^{17} \\ &+ (q-1)(40q^5+1193q^4+\ldots)u^{18} \\ &+ (q-1)(216q^5+\ldots)u^{19} + (q-1)(22q^6+\ldots)u^{20}. \end{aligned} \tag{5}$$

This expansion is based on that of Kihara *et al* (1954) with minor corrections to the coefficient of u^{16} and new terms given for higher powers of u. If the number of q-colourings of the square lattice of N sites tends to W^N then using the relation

$$W = (q-1)^2/q \times \Lambda(u = -1/q - 1, q, \mu = 1) \tag{6}$$

reduces the expansion (5) to

$$W = [(q-1)^2/q](1 + x^3 + x^7 + 3x^8 + 4x^9 + 3x^{10} + 3x^{11} + 11x^{12} + 24x^{13} + \ldots) \tag{7}$$

with

$$x = 1/(q-1). \tag{8}$$

(In general the q-colouring expansions must be obtained from high temperature expansions but for the square lattice, high temperature and low temperature expansions are equivalent because of the duality relation (Potts 1952, Kihara *et al* 1954).) The final two terms of (7) disagree with those given by Nagle (1971) but agree with terms calculated by Kim and Enting (1977), using two independent techniques. The other series that indicated errors in the Kihara *et al* series is the $q = 3$ free energy series calculated by de Neef (1975). The necessary corrections to the u^{16} coefficient can be determined once the higher-order terms in (5) are known together with the series and the series for $q = 2$ and $q = 3$.

A consistency check on this calculation of the correction terms incorporated in (5) is obtained by considering the $q = 1$ case where

$$\frac{\partial}{\partial q}(\Lambda(u, q, \mu = 1))\bigg|_{q=1} = \lim_{q \to 1} (\ln \Lambda(u, q, \mu = 1))/(q-1). \tag{9}$$

This expression corresponds to the mean number of clusters in a bond percolation problem. Series can be obtained by using the method of partial generating functions (Sykes *et al* 1965) as generalized to the Potts model by Enting (1974a, b, 1975).

The fact that the Kihara *et al* series are in error for $q = 3$ indicates that there is a mistake in the $q = 3$ expression for $\psi_{16}(\mu, 3)$ given by Enting (1974b). The corrected expression is

$$\psi_{16}(\mu, 3) = 2\mu^{16} + 12\mu^{15} + 44\mu^{14} + 136\mu^{13} + 318\mu^{12} + 716\mu^{11} + 1272\mu^{10} + 1982\mu^{9} + 1788\mu^{8} + 44\mu^{7} - 5580\mu^{6} + 2670\mu^{5} - 836\mu^{4}.$$

The importance of these corrections stems from the use of these series in cross checking of special cases. The series obtained by de Neef (1975) and by Kim and Enting include more terms than are given here. Comparison of all previously known terms against the new series is an important test of the accuracy of the derivation of these new series. The analysis of these extended series is to be published elsewhere in the context of the particular special cases.

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